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Generalization of the Gross-Perry Metrics

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Abstract A class of $SO(n+1)$ symmetric solutions of the $(N+n+1)$-dimensional Einstein equations is found. It contains 5-dimensional metrics of Gross and Perry and Millward.

Keywords Higher dimensional gravity · Gross-Perry metrics · Exact solutions

1 Introduction

An extra spatial dimension was introduced by Kaluza and Klein (see e.g. [5]) in order to unify electromagnetism and gravity. Recently, due to the string theory, extra dimensions became a permanent part of the high energy theoretical physics. For instance, in brane-world models (see [11] for a review) matter fields are confined to a four-dimensional brane and gravity can propagate in higher dimensional bulk. These higher dimensional models motivate studying Einstein's equations in $D > 4$ dimensions. Some techniques of the 4-dimensional Einstein theory were already generalized to higher dimensions. They refer mainly to the classification of the Weyl tensor, the Robinson-Trautman solutions, space-times with vanishing invariants and metrics with D-2 abelian symmetries (see e.g. [3, 4, 9, 14]).

Symmetry assumptions are one the most efficient methods of solving the Einstein equations. All well known multidimensional exact solutions like the Myers-Perry black hole [13], black ring of Emparan and Reall [7] and the Gross-Perry metrics [8] (see also [6]) admit several dimensional symmetry groups.

In [10] we proposed a construction of vacuum metrics admitting $SO(n+1)$ spherical symmetry, which was based on the symmetry reduction of $(N+n+1)$-dimensional Einstein equations to $(N+1)$-dimensional equations with a scalar field ϕ. There was used an additional assumption that the field of normal vectors to surfaces $\phi = const$ is geodetic and

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the induced metric of the surfaces is an Einstein metric. The construction, for zero cosmological constant and timelike surfaces $\phi = const$, can be summarized as follows.

Let γ_{ij} and P_{ij} be symmetric tensors depending on coordinates x^i, $i = 0, \ldots, N-1$. Assume that γ_{ij} has the Lorentz signature and P_{ij} satisfies the following conditions

$$P^i_{\ i} = 0, \tag{1}$$

$$P^i_{\ j} P^j_{\ i} = 2c = const, \tag{2}$$

$$P^k_{\ i;k} = 0, \tag{3}$$

where $P^i_{\ j} = \gamma^{ik} P_{ij}$ and the semicolon denotes covariant derivative related to the metric $\gamma_{ij} dx^i dx^j$. From matrices $\gamma = (\gamma_{ij})$ and $P = (P^i_{\ j})$ we compose metric corresponding to $\gamma e^{P\tau}$, where τ is a function of another coordinate s. We assume that its Ricci tensor satisfies

$$R^i_{\ j}(\gamma e^{P\tau}) = \lambda \delta^i_{\ j}, \quad \lambda = const. \tag{4}$$

Given c and λ we look for solutions $\beta(s)$ and $\phi(s)$ of the following equations

$$(\beta\dot{\phi})^{\cdot} = -\beta V_{,\phi}, \tag{5}$$

$$-N\lambda\beta^{-2/N} = \left(1 - \frac{1}{N}\right)\frac{\dot{\beta}^2}{\beta^2} - \frac{2c}{\beta^2} - \dot{\phi}^2 + 2V, \tag{6}$$

where the dot denotes the partial derivative with respect to s and V is a function of ϕ.

The main result of [10] is that, under conditions (1)–(6), metric

$$\tilde{g} = -ds^2 + \tilde{g}_{ij} dx^i dx^j, \tag{7}$$

where

$$\tilde{g}_{ij} = \beta^{2/N} (\gamma e^{P\tau})_{ij} \tag{8}$$

and $\tau(s)$ is defined via equation

$$\beta\dot{\tau} = 2, \tag{9}$$

satisfies $(N+1)$-dimensional Einstein equations with the scalar field ϕ and potential V. Moreover, if

$$V = -\frac{1}{2} n\,(n-1)\, e^{-2\sqrt{\frac{n+N-1}{n(N-1)}}\phi} \tag{10}$$

then

$$g = e^{-2\sqrt{\frac{n}{(N-1)(n+N-1)}}\phi}\, \tilde{g} - e^{2\sqrt{\frac{N-1}{n(n+N-1)}}\phi} d\Omega_n^2 \tag{11}$$

is an $(N+n+1)$-dimensional vacuum metric invariant under the group $SO(n+1)$. Here $d\Omega_n^2$ is the standard metric of the n-dimensional sphere.

A particular solution of conditions (1)–(4), for any $N > 1$, is given by

$$\gamma_{ij} = \mathrm{diag}(+1, -1, -1, \ldots), \qquad P_{ij} = P_{ji} = const, \qquad P^i_{\ i} = 0, \qquad \lambda = 0. \tag{12}$$

For $N=2$ conditions (1)–(4) can be solved in full generality. They lead either to (12) or to $c=0$ and to $\gamma e^{P\tau}$ equivalent to the metric

$$(\gamma e^{P\tau})_{ij}dx^i dx^j = \frac{du dv}{(1+\frac{\lambda}{4}uv)^2} + \tau h(u) du^2, \tag{13}$$

where h is an arbitrary function of coordinate u. In the next section we find solutions of equations (5), (6) and construct corresponding vacuum metrics. In Sect. 3 we discuss properties of these metrics.

2 Multi Dimensional Vacuum Metrics

In [10] we gave examples of vacuum metrics derived by our method. Other solutions with $n>1$ can be obtained by inspection of the Gross-Perry metrics [8]. Let $\lambda=0$ and $s=s(r)$ be a function of a new coordinate r. Then equations (5), (6), (9) take the form

$$\left(\frac{\beta\phi'}{\alpha}\right)' = \alpha\beta V_{,\phi} \tag{14}$$

$$\left(1-\frac{1}{N}\right)\frac{\beta'^2}{\alpha^2} - \frac{\beta^2\phi'^2}{\alpha^2} + 2\beta^2 V = 2c, \tag{15}$$

$$\tau' = \frac{2\alpha}{\beta}, \tag{16}$$

where the prime denotes the derivative with respect to r and $\alpha = s'$. Metric (7) is given by

$$\tilde{g} = -\alpha^2 dr^2 + \beta^{2/N}(\gamma e^{P\tau})_{ij}dx^i dx^j. \tag{17}$$

If $N=n=2$ equations (14), (15) are satisfied by functions α, β, ϕ corresponding to the Gross-Perry metric [8]. Changing parameters in these functions leads to the following solutions for arbitrary dimensions $N>1$ and $n>1$

$$\alpha = \alpha_0|r|^{-l-1}|r-r_0|^{l-p}|r+r_0|^{l+p} \tag{18}$$

$$\beta = \beta_0(r^2-r_0^2)\alpha \tag{19}$$

$$e^{\sqrt{\frac{n+N-1}{n(N-1)}}\phi} = (n-1)|r\alpha|. \tag{20}$$

Here l is a number defined by n and N

$$l = \frac{n+N-1}{(n-1)(N-1)} \tag{21}$$

and p, α_0, β_0 and $r_0 \neq 0$ are parameters related to the constant c via

$$c = 2\beta_0^2 r_0^2\left[\frac{n}{n-1} - p^2\frac{(n-1)(N-1)^2}{N(n+N-1)}\right]. \tag{22}$$

Integrating (16) yields

$$\tau = \frac{1}{\beta_0 r_0}\ln\left|\frac{r+r_0}{r-r_0}\right| + \tau_0. \tag{23}$$

Due to a freedom of transformations of r, P and γ we can assume

$$r_0 > 0, \qquad |\alpha_0| = \frac{1}{n-1}, \qquad \beta_0 = 1, \qquad \tau_0 = 0 \tag{24}$$

(note that a sign of α_0 can be still adjusted to have $\beta > 0$ for $r \neq 0, \pm r_0$). Thus, p and $r_0 > 0$ remain as free parameters.

Let $N = 2$. In the case (12) and $c > 0$ the matrix P can be diagonalized by a 2-dimensional Lorentz transformation. Hence, one obtains

$$\tilde{g} = -\alpha^2 dr^2 + \beta\left(e^{\pm\tau\sqrt{c}}dt^2 - e^{\mp\tau\sqrt{c}}dy^2\right), \tag{25}$$

where t and y denote coordinates x^i. Substituting (18)–(24) into (25) and (11) yields the following $(n+3)$-dimensional vacuum metric

$$g = \left|\frac{r-r_0}{r+r_0}\right|^{p'-q} dt^2 - \left|\frac{r-r_0}{r+r_0}\right|^{p'+q} dy^2 - \frac{|r+r_0|^{\frac{2p'+2}{n-1}}}{|r|^{\frac{2n}{n-1}}|r-r_0|^{\frac{2p'-2}{n-1}}}\left(\frac{dr^2}{(n-1)^2} + r^2 d\Omega_n^2\right). \tag{26}$$

Parameters p' and q are related to p and c by

$$p' = \frac{n-1}{n+1}p, \qquad q = \pm\frac{\sqrt{|c|}}{r_0}. \tag{27}$$

Because of (22) they are constrained by

$$(n+1)p'^2 + (n-1)q^2 = 2n. \tag{28}$$

For $n = 2$ solution (26) is exactly the Gross-Perry metric [8] under the identification

$$r_0 = m, \qquad p' = \frac{1}{\alpha}(\beta+1), \qquad q = \frac{1}{\alpha}(\beta-1). \tag{29}$$

Here m, α and β are parameters used by Gross and Perry, constrained by the condition $\alpha = \sqrt{\beta^2+\beta+1}$.

If $c < 0$ the matrix P can be put into the off diagonal form. Instead of (25) one obtains

$$\tilde{g} = -\alpha^2 dr^2 + \beta\left[\cos(\tau\sqrt{|c|})(dt^2 - dy^2) \pm 2\sin(\tau\sqrt{|c|})dtdy\right]. \tag{30}$$

In this case the vacuum metric corresponding to (18)–(24) reads

$$\begin{aligned} g = {}& \left|\frac{r-r_0}{r+r_0}\right|^{p'}\left[\cos\left(q\ln\left|\frac{r+r_0}{r-r_0}\right|\right)(dt^2-dy^2) + 2\sin\left(q\ln\left|\frac{r+r_0}{r-r_0}\right|\right)dtdy\right] \\ & - \frac{|r+r_0|^{\frac{2p'+2}{n-1}}}{|r|^{\frac{2n}{n-1}}|r-r_0|^{\frac{2p'-2}{n-1}}}\left(\frac{dr^2}{(n-1)^2} + r^2 d\Omega_n^2\right). \end{aligned} \tag{31}$$

Relation (27) is still valid, but now parameters p', q are constrained by

$$(n+1)p'^2 - (n-1)q^2 = 2n. \tag{32}$$

If $N=2$ and

$$p=\pm\frac{\sqrt{2n(n+1)}}{n-1} \tag{33}$$

then it follows from (22) that $c=0$ and one can merge solutions (18)–(20) with metric (13) for $\lambda=0$. In this way the following vacuum metric is obtained

$$\begin{aligned} g=&\left|\frac{r-r_0}{r+r_0}\right|^{\pm\sqrt{\frac{2n}{n+1}}}\left(dudv+\ln\left|\frac{r+r_0}{r-r_0}\right|h(u)du^2\right)\\ &-\frac{|r+r_0|^{\frac{2}{n-1}(\pm\sqrt{\frac{2n}{n+1}}+1)}}{|r|^{\frac{2n}{n-1}}|r-r_0|^{\frac{2}{n-1}(\pm\sqrt{\frac{2n}{n+1}}-1)}}\left(\frac{dr^2}{(n-1)^2}+r^2d\Omega_n^2\right). \end{aligned} \tag{34}$$

In the case $n=2$, $h(u)=0$ metric (34) with the lower sign coincides with the metric given by Millward [12] under the identification

$$b=\frac{1}{\sqrt{3}}\ln\left|\frac{r-r_0}{r+r_0}\right|, \qquad M=\frac{\sqrt{3}}{2}r_0. \tag{35}$$

For $N>2$ one can easily construct vacuum solutions based on relations (11), (12), (17)–(24). They generalize metrics (26) and (31). In this case a classification of symmetric tensors (here P_{ij}) in multidimensional Lorentzian manifolds [3] can be useful in order to distinguish nonequivalent solutions. One can also construct metrics which generalize (34) by taking $\gamma e^{P\tau}$ corresponding to the metric

$$dudv+\tau h(u)du^2+\sum_{a=1}^{N-2}e^{c_a\tau}dy_a^2, \tag{36}$$

where constants c_a are constrained by

$$\sum_{a=1}^{N-2}c_a=0. \tag{37}$$

In this case we can use functions defined by (18)–(24) with constant c given by

$$c=\frac{1}{2}\sum_{a=1}^{N-2}c_a^2. \tag{38}$$

3 Discussion

In addition to $SO(n+1)$ symmetries metrics (26) and (31) admit one timelike and one spacelike Killing vector (note that interpretation of ∂_t and ∂_y in case (31) can change depending on value of r). Metric (26) is static and metric (31) is stationary. In the limit $r\to\infty$ they behave like

$$dt^2-dy^2-r^{-2\frac{n-2}{n-1}}\left(\frac{dr^2}{(n-1)^2}+r^2d\Omega_n^2\right). \tag{39}$$

Under the change $r' = r^{\frac{1}{n-1}}$ metric (39) takes the standard form of the $(n+3)$-dimensional Minkowski metric. Thus, metrics (26) and (31) are asymptotically flat on surfaces $y = const$.

Metric (34) has a null Killing vector field ∂_v and it belongs to generalized Kundt's class [1]. If $r \to \infty$ it tends asymptotically to the flat metric in the form

$$dudv - r^{-2\frac{n-2}{n-1}} \left(\frac{dr^2}{(n-1)^2} + r^2 d\Omega_n^2 \right). \tag{40}$$

Generalizing results of [2] for the Gross-Perry metric to arbitrary n, one can show that both metrics (26) and (31) are of algebraic type I. For $h \neq 0$ metric (34) is of algebraic type II_i and for $h(u) = 0$ it is of type D. Aligned null vector fields for metrics (26), (31) and (34) are given in Appendix A.

All metrics (26), (31) and (34) are singular at $r = \pm r_0$ and $r = 0$. Near $r = 0$ they behave as

$$dt^2 - dy^2 - r^{-\frac{2n}{n-1}} \left(\frac{dr^2}{(n-1)^2} + r^2 d\Omega_n^2 \right). \tag{41}$$

Substituting $r' = r^{-\frac{1}{n-1}}$ shows that (41) is the flat metric. Thus, $r = 0$ is a coordinate singularity. By calculating the Kretschmann invariant (see Appendix B) it can be shown, that singularity at $r = \pm r_0$ is essential for all values of parameters in the case of metrics (26) and (34). In the case of metric (31) the singularity at $r = r_0$ and $r = -r_0$ is essential when, respectively, $p' < n$ or $p' > -n$. For $p' > n$ or $p' < -n$ the geodesic distance along ∂_r tends to infinity when $r \to r_0$ or $r \to -r_0$, respectively. Thus, these regions represents an infinity different from that given by $r \to \infty$. For these values of parameters the Riemann tensor (in an orthonormal basis) tends to zero when $r \to r_0$ or $r \to -r_0$, respectively. However, the asymptotic metric is not the $(n+3)$-dimensional Minkowski metric. Its coefficients in front of dt and dy tend to zero whereas the coefficient in front of $d\Omega_n^2$ tends to infinity like the geodesic distance to the power $2(p'-1)/(p'-n)$ or $2(p'+1)/(p'+n)$, respectively.

Since metrics (26), (31) are invariant under the nonnull field ∂_u they can be interpreted in the context of the Kaluza-Klein theory. Then metric (26) is equivalent to the scalar field given by g_{yy} and the asymptotically flat $(n+2)$-dimensional metric induced on the surface $y = const$. The case $n = 2$ (the Gross-Perry metric) was studied in this framework by Ponce de Leon [15]. In order to interpret metric (31) with $q \neq 0$ in this vein one can write it in the form

$$g = -\Phi(dy - A_0 dt)^2 + g_{n+2}. \tag{42}$$

Here

$$A_0 = \tan\left(q \ln \left| \frac{r + r_0}{r - r_0} \right| \right) \tag{43}$$

is the electromagnetic potential,

$$\Phi = \left| \frac{r - r_0}{r + r_0} \right|^{p'} \cos\left(q \ln \left| \frac{r + r_0}{r - r_0} \right| \right) \tag{44}$$

corresponds to a scalar field and

$$g_{n+2} = \left| \frac{r - r_0}{r + r_0} \right|^{p'} \frac{dt^2}{\cos\left(q \ln \left|\frac{r+r_0}{r-r_0}\right|\right)} - \frac{|r + r_0|^{\frac{2p'+2}{n-1}}}{|r|^{\frac{2n}{n-1}} |r - r_0|^{\frac{2p'-2}{n-1}}} \left(\frac{dr^2}{(n-1)^2} + r^2 d\Omega_n^2 \right) \tag{45}$$

defines, modulo a power of Φ, a $(n+2)$-dimensional metric. This metric is Lorentzian and asymptotically flat for large values of r and becomes singular when r diminishes to a value satisfying condition $q\ln|\frac{r+r_0}{r-r_0}|=\pm\pi/2$.

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Appendix A

The aligned null direction for metric (26) is given by

$$\hat{l}=\left(\left|\frac{r-r_0}{r+r_0}\right|^{p'-q}+\frac{1}{2}f^2\right)dt+\left(\left|\frac{r-r_0}{r+r_0}\right|^{p'}-\frac{1}{2}f^2\left|\frac{rr_0}{r+r_0}\right|^{q}\right)dy$$
$$+f\frac{|r+r_0|^{\frac{p'+1}{n-1}}}{(n-1)|r|^{\frac{n}{n-1}}|r-r_0|^{\frac{p'-1}{n-1}}}dr, \tag{A.1}$$

and for the mertric (31)

$$\hat{l}=\left(\sin\left(q\ln\left|\frac{r+r_0}{rr_0}\right|\right)+1\right)\left(1-\frac{1}{2}f^2\left|\frac{rr_0}{r+r_0}\right|^{p'}\frac{\cos\left(q\ln|\frac{r+r_0}{rr_0}|\right)}{(\sin\left(q\ln|\frac{r+r_0}{rr_0}|\right)+1)^2}\right)dt$$
$$+\left(\frac{1}{2}f^2\left|\frac{r-r_0}{r+r_0}\right|^{p'}-\cos\left(q\ln\left|\frac{r+r_0}{rr_0}\right|\right)\right)dy+f\frac{|r+r_0|^{\frac{p'+1}{n-1}}}{(n-1)|r|^{\frac{n}{n-1}}|r-r_0|^{\frac{p'-1}{n-1}}}dr. \tag{A.2}$$

Function f is a solution of the polynomial equation

$$f^4-8f^2\left(\frac{p'}{q}+\frac{2q(n-1)rr_0}{n(r^2+r_0^2-2p'rr_0)}\right)A-16A^2=0, \tag{A.3}$$

where

$$A=\left|\frac{r-r_0}{r+r_0}\right|^{p'-q} \tag{A.4}$$

in the case (26) and

$$A=\left|\frac{r-r_0}{r+r_0}\right|^{-p'}\left(\sin\left(q\ln\left|\frac{r+r_0}{r-r_0}\right|\right)+1\right). \tag{A.5}$$

in the case (31). For the metric (34) the aligned null directions are following

$$\hat{n}=\left|\frac{r+r_0}{r-r_0}\right|^{\pm\sqrt{\frac{2n}{n+1}}}du, \tag{A.6}$$

$$\hat{l}=\frac{1}{2}\left(\ln\left|\frac{rr_0}{r+r_0}\right|+f^2\left|\frac{r+r_0}{rr_0}\right|^{\pm\sqrt{\frac{2n}{n+1}}}\right)du+\frac{1}{2}dv$$
$$+\frac{|r+r_0|^{\frac{1}{n-1}(\pm\sqrt{\frac{2n}{n+1}}+1)}}{(n-1)|r|^{\frac{n}{n-1}}|r-r_0|^{\frac{1}{n-1}(\pm\sqrt{\frac{2n}{n+1}}-1)}}dy. \tag{A.7}$$

In this case

$$f^2 = \left(\mp\sqrt{\frac{n+1}{2n}} + \frac{2(n^2-1)rr_0}{n(n+1)(r^2+r_0^2) \mp \sqrt{2n(n+1)}rr_0} \right) h(u) \left| \frac{r+r_0}{r-r_0} \right|^{\mp\sqrt{\frac{2n}{n+1}}}. \quad \text{(A.8)}$$

Appendix B

The Kretschmann invariant for metrics (26) and (31) has the following form

$$\begin{aligned} R_{\mu\nu\delta\sigma} R^{\mu\nu\delta\sigma} &= 16n(n-1)r_0^2 |r|^{\frac{2(n+1)}{n-1}} |r - r0|^{-\frac{4(np')}{n-1}} |r + r0|^{-\frac{4(n+p')}{n-1}} \Big((n^2+n-2p'^2)r^4 \\ &\quad + 2p'(n(n+1)(p'^2-3) + 2(1+p'^2))r^3 r_0 \\ &\quad - (4-3n-5n^2-2(-2+n(n+3))p'^2 + (4+n(n+3))p'^4)r^2 r_0^2 \\ &\quad + 2p'(n(n+1)(p'^2-3) + 2(1+p'^2))r r_0^3 + (n^2+n-2p'^2)r_0^4 \Big) \end{aligned} \quad \text{(B.1)}$$

In the case of metric (34) the Kretschmann invariant has the form (B.1) with $p' = \sqrt{\frac{2n}{n+1}}$.

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